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FAST TRACK COMMUNICATION

Construction of the $\mathcal{C}$ operator for a $\mathcal{PT}$ symmetric model

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Abstract
We obtain a closed form expression of the $\mathcal{C}(x, y)$ operator for the $\mathcal{PT}$ symmetric Scarf I potential. It is also shown that the eigenfunctions form a complete set.

In recent years non-Hermitian systems, in particular the $\mathcal{PT}$ symmetric ones [1] have been studied widely. Many of these systems are characterized by the fact that they possess real eigenvalues. However for non-Hermitian systems the concept of a scalar product is a non-trivial one. In fact a straightforward $\mathcal{PT}$ symmetric generalization of the usual scalar product for Hermitian systems produces a norm which alternates in sign i.e.,

$$\langle\psi_m|\psi_n\rangle_{\mathcal{PT}} = (-1)^n\delta_{mn}. \tag{1}$$

With a view to circumvent this difficulty an operator $\mathcal{C}(x, y)$ was introduced [2]. This operator is defined as [2]

$$\mathcal{C}(x, y) = \sum_{n=0}^{\infty}\psi_n(x)\psi_n(y) \tag{2}$$

where $\psi_n(x)$ are eigenfunctions of the Hamiltonian H:

$$H\psi_n(x) = \lambda_n\psi_n(x). \tag{3}$$

However, it is not always easy to obtain a closed form expression of the $\mathcal{C}(x, y)$ operator and often one has to construct it using various approximating techniques [3]. Our purpose here is to obtain a closed form expression of the $\mathcal{C}(x, y)$ operator for the $\mathcal{PT}$ symmetric Scarf I potential.

We consider the Scarf I potential defined by

$$V(x) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)\frac{1}{\cos^2 x} + \frac{\alpha^2-\beta^2}{2}\frac{\sin x}{\cos^2 x}, \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{4}$$

where α and β are complex parameters such that $\beta^* = \alpha$ and $\alpha_R > \frac{1}{2}$. In this case the (real) eigenvalues and the corresponding eigenfunctions are given by [5]

$$E_n = \left(n + \frac{\alpha+\beta+1}{2}\right)^2$$
$$\psi_n(x) = D_n(1-\sin x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+\sin x)^{\frac{\alpha^*}{2}+\frac{1}{4}} P_n^{(\alpha,\alpha^*)}(\sin x), \qquad n = 0, 1, 2, \ldots, \tag{5}$$

where $P_n^{(a,b)}(x)$ denotes the Jacobi polynomial and D_n is a normalization constant given by

$$D_n = i^n \sqrt{\frac{(2n+2\alpha_R+1)n!\Gamma(n+2\alpha_R+1)}{2^{2\alpha_R+1}\Gamma(n+\alpha+1)\Gamma(n+\alpha^*+1)}}. \tag{6}$$

Using the orthogonality properties of Jacobi polynomials [4] it can be shown [5] that the wavefunctions in (5) satisfy the relation

$$\int_{-\pi/2}^{\pi/2} (\mathcal{PT}\psi_m(x))\psi_n(x)\,\mathrm{d}x = (-1)^n \delta_{mn}. \tag{7}$$

We now turn to the evaluation of the $\mathcal{C}(x, y)$ operator. Using (5) we obtain from (2)

$$\mathcal{C}(x,y) = \prod_{z=x,y}(1-\sin z)^{\frac{\alpha}{2}+\frac{1}{4}}(1+\sin z)^{\frac{\alpha^*}{2}+\frac{1}{4}}$$
$$\times \sum_{n=0}^{\infty} \frac{(-1)^n(2n+2\alpha_R+1)n!\Gamma(n+2\alpha_R+1)}{2^{2\alpha_R+1}\Gamma(n+\alpha+1)\Gamma(n+\alpha^*+1)} P_n^{(\alpha,\alpha^*)}(\sin x) P_n^{(\alpha,\alpha^*)}(\sin y). \tag{8}$$

To evaluate the summation in (8) we now use the result [6]

$$\sum_{n=0}^{\infty} n! \frac{(2\alpha_R+1)_n}{(\alpha+1)_n(\beta+1)_n}(2n+2\alpha_R+1) P_n^{(\alpha,\alpha^*)}(\sin x) P_n^{(\alpha,\alpha^*)}(\sin y) t^n$$
$$= \frac{(2\alpha_R+1)(1-t)}{(1+t)^{2\alpha_R+1}} F_4(a,b,c,d,U,V) \tag{9}$$

where

$$F_4(a,b,c,d,U,V) = \sum_{r,s=0}^{\infty} \frac{(a)_s(b)_s}{s!(d)_s} \frac{(a+s)_r(b+s)_r}{r!(c)_r} U^r V^s$$
$$= \sum_{s=0}^{\infty} \frac{(a)_s(b)_s V^s}{s!(d)_s} {}_2F_1(a+s, b+s, c, U) \tag{10}$$

$$a = \alpha_R + 1, \qquad b = \alpha_R + 3/2, \qquad c = 1+\alpha, \qquad d = \beta + 1,$$
$$U = \frac{(1-\sin x)(1-\sin y)t}{(1+t)^2}, \qquad V = \frac{(1+\sin x)(1+\sin y)t}{(1+t)^2} \tag{11}$$

and ${}_2F_1(a, b, c, z)$ is the standard hypergeometric function.

Now taking the limit $t \to -1$, we obtain

$$F_4(a,b,c,d,U,V) = (-U)^a \sum_{s=0}^{\infty} \frac{\Gamma(c)\Gamma(1/2)}{\Gamma(b+s)\Gamma(c-a-s)} \frac{(-V/U)^s(a)_s(b)_s}{s!(d)_s}. \tag{12}$$

Then using (12) we obtain from (9) and (10)

$$\mathcal{C}(x,y) = \mathcal{N}\frac{[(1+\sin x)(1+\sin y)]^{(\alpha^*/2+1/4)}}{[(1-\sin x)(1-\sin y)]^{(\alpha^*/2+3/4)}} {}_2F_1(a, 1-c+b, d, z),$$
$$z = \frac{(1+\sin x)(1+\sin y)}{(1-\sin x)(1-\sin y)} \tag{13}$$

where $\mathcal{N}$ is a constant given by

$$\mathcal{N} = \frac{2\Gamma(\alpha_R + 1)\sin(\pi(1 - c + a))\Gamma(1 - c + a)}{\pi\Gamma(\alpha^* + 1)}. \tag{14}$$

It may be noted that (13) is an exact result.

Completeness of the eigenfunctions

The completeness property is a very important feature of eigenfunctions. However, to the best of our knowledge for $\mathcal{PT}$ symmetric systems this property has been verified numerically [7]. Here we shall show analytically that the eigenfunctions (5) form a complete set. To do this, we note that in a $\mathcal{PT}$ symmetric theory with unbroken $\mathcal{PT}$ symmetry the completeness property can be expressed as [2, 3]

$$\sum_{n=0}^{\infty}(-1)^n\psi_n(x)\psi_n(y) = \delta(x - y). \tag{15}$$

To prove (15) we consider the result [8]

$$\sum_{n=0}^{\infty}\frac{n!\Gamma(a+b+2n+1)\Gamma(a+b+n+1)}{\Gamma(a+n+1)\Gamma(b+n+1)}P_n^{(a,b)}(x)P_n^{(a,b)}(y)$$
$$= (1+x)^{-b/2}(1-x)^{-a/2}(1+y)^{-b/2}(1-y)^{-a/2}\delta(x-y) \tag{16}$$

where $-1 < x, y < 1, \mathrm{Re}(a) > -1, \mathrm{Re}(b) > -1$. Now putting $a = \alpha, b = \beta$ in (16) and using (5) we obtain

$$\sum_{n=0}^{\infty}(-1)^n\psi_n(x)\psi_n(y) = \sqrt{\cos x\cos y}\,\delta(\sin x - \sin y) = \delta(x - y). \tag{17}$$

Thus the eigenfunctions (5) form a complete set.

It is interesting to note that two important results can be derived using (16). First we recall that in Hermitian systems, the operator $\mathcal{C}(x, y)$ is just the parity operator i.e., $\mathcal{C}(x, y) = \delta(x + y)$. So for $\alpha = \alpha^*$, (13) should reduce to this limit. Now using the properties of hypergeometric functions it can be shown that for real $\alpha, \beta, \mathcal{C}(x, y) = \delta(x + y)$. The other properties of the $\mathcal{C}$ operator namely, $\mathcal{C}\psi_n = (-1)^n\psi_n$ follow from the definition (2) and (1) while $\mathcal{C}^2 = 1$ can be derived using the results (7) and (16).

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Corrigendum

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Equation (9) should read as

$$\sum_{n=0}^{\infty} n!\frac{(2\alpha_R+1)_n}{(\alpha+1)_n(\beta+1)_n}(2n+2\alpha_R+1)P_n^{(\alpha,\alpha^*)}(\sin x)P_n^{(\alpha,\alpha^*)}(\sin y)t^n = \frac{(2\alpha_R+1)(1-t)}{(1+t)^{2\alpha_R+2}}\ F_4(a,b,c,d,U,V)$$

Equation (13) should read as

$$\mathcal{C}(x,y) = \mathcal{N}\frac{[(1+\sin x)(1+\sin y)]^{(\alpha^*/2+1/4)}}{[(1-\sin x)(1-\sin y)]^{(\alpha^*/2+3/4)}}\,{}_2F_1(a,1-c+a,d,z), \quad z=\frac{(1+\sin x)(1+\sin y)}{(1-\sin x)(1-\sin y)}.$$

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